An Economic Approach to the Optimal Purchase of Armor, Health, and Magic Resistance

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1 An Introduction to Utility and Effective Health

Given two goods, armor and health (equivalently, magic resistance and health), economics dictates that there should be some point where we can maximize the utility given by each good subject to the constraint of how much money we have. This is known as utility maximization. In simpler terms, we will attempt to show that, given some amount of gold, there is an optimal amount of both armor and health (alternatively, magic resistance and health) that will provide the most utility to the player.

What is utility? Utility is, informally, the perceived usefulness of an item. While there is no such thing as intrinsic utility, it is assumed that, by buying armor, health, and magic resistance, the player desires to remain alive during the game. As such, we shall assume that the utility of armor, health, and magic resistance is such that they help the player to stay alive.

Once a player's health bar reaches 0, the player dies. That, for the most part, is bad - by dying, the player sacrifices many opportunities to progress in game. The player cannot farm minions, contribute to team fights, apply pressure on the map, etc. However, health is a modified resource. Both armor and magic resistance *modify* health by making it harder for enemies to take away points of health. By mitigating the amount of damage a player takes (and thus the amount of points of health removed), armor and magic resist scale the amount of health a player has. This amount of scaled health is called *effective health*. Effective health is defined as the amount of damage a player must take to remove all points of health from their health bar after considerations of damage mitigation have been taken into account. Consider the following function that calculates the damage multiplier on physical damage at different levels of armor.[1]

$$\text{Physical Damage Multiplier} = \begin{cases} \frac{100}{100+\text{armor}} & : \text{armor} \geq 0 \\ 2 - \frac{100}{100+\text{armor}} & : \text{armor} < 0 \end{cases}$$

Armor and magic resistance scale health by reducing the amount of damage taken from enemy sources. These resistances apply before damage is applied, fewer points of health will be taken from the player's champion. Mathematically, resistances scale in a $1 : 0.01$ ratio with health. Namely, each additional point of armor or magic resistance will increase effective health by $\approx 1\%$. While the percentage is not exact due to rounding, plugging in a few values (50, 75, 100, 200) will show how health scales with armor.

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[1] This formula is equivalent for magic resistance and magic damage.

For 50 points of armor, there will be a damage multiplier of $\frac{2}{3}$. So, if the player has x health, then to kill them, they must must take $\frac{2}{3} *$ Damage Dealt $= x$. Thus, Damage Dealt $= \frac{3}{2} * x$. $\left(\frac{3}{2} - 1\right)/100 = 50\%$. Thus, 50 armor scales health by 50%. This is true for all values of both armor and magic resist.[2]

In discussing purchases of different amounts of armor, health, and magic resistance, then, we will treat effective health as a good proxy for utility. Why? First, more effective health will keep the player alive longer (which we have assumed is a good thing). This is known as monotonicity of preferences where players will always prefer more effective health to less. Second, we are only considering binary choices of health and armor or health and magic resistance. There is no consideration of buying damage items, wards, or elixirs. Further, there is no consideration given to the utility of different items - Sunfire Cape might provide less armor and health than Randuin's, but it has a damage per second aura that may be useful. Thus, maximizing effective health to stay alive longer will be the only consideration in our model - this makes it a good proxy for utility.

As a final note, we will begin by only considering binary relationships between health and armor and health and magic resistance. Since enemy team compositions will vary drastically, our model would not be very accurate without that information. For example, it is not a good purchase to buy magic resistance against a full attack damage team composition as magic resistance would not contribute to effective health against physical damage.

2 Allocating a Budget to Optimal Amounts of Armor, Health, and Magic Resistance

This next section will be somewhat more technical. First, we will describe the budget constraint. Formally, the budget constraint is given by the following equation:

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[2] Check the League of Legends Wiki at http://leagueoflegends.wikia.com/wiki/Armor for a further discussion of the scaling of resists.

$$B \geq p_x x + p_y y$$

The budget constraint is all "bundles" of 2 goods such that the price of the bundle is less than the amount of money B. A bundle is simply a combination of two goods. In this equation, p_x is the price of good x, and x is the amount of good x. This is similar for p_y and y.

In League of Legends terms, the budget constraint would be a combination of armor and health (or magic resistance and health) whose price to purchase is less than the amount of gold a player has.

Continuing this line of reasoning, we wish to optimally allocate our gold such that we can purchase the most effective health at different gold levels. Take the following example:

> A player is playing Dr. Mundo. The enemy team is composed of Riven, Zed, Xin Zhao, Graves, and Taric. Assuming he only builds tank items, what build path should he take at 2000 gold? 5000 gold? $10,000$ gold?
>
> Next, Mundo has been following his optimal build path and has a certain amount of gold x. What is the best way he can spend x gold to maximize his tanking power?

In this example, Mundo has many choices. He can buy a whole host of items, but if he does so inefficiently, his tanking power will be wasted. He will have spent his gold inefficiently, not only hurting himself but his team that relies on him. It is not immediately clear, though, which options are the best, especially if we disregard the utility that different items provide.

Thus, we wish to maximize Mundo's effective health so that he can stay alive longer, more effectively tank for his team, and hopefully contribute to winning the game. To do this, we need to maximize the utility function

$$U(x, y) \text{ subject to the constraint } B = p_x x + p_y y$$

Remember, since we wish to maximize the amount of effective health a player's champion has, we will only consider bundles that lie on or near the budget constraint.

From this, we can derive the Marginal Rate of Substitution or the rate at which a player should be willing to give up a unit of armor compared to a unit of health.

$$\text{MRS} = \frac{p_x}{p_y}$$

The Marginal Rate of Substitution between armor and health is just the ratio of their per-unit costs. While these are not always the same, we will take the weighted average of the basic health items and basic armor items to come up with a relatively good measure of per-unit prices. These come out to be 2.6 gold per point of health, 19.4 gold per unit of armor, and 19.4 gold per unit of magic resistance. Thus, our Marginal Rate of Substitution for armor and health (magic resistance and health) is:

$$\text{MRS} = \frac{p_x}{p_y} = \frac{2.6}{19.4} \approx 0.134$$

This essentially shows that a player ought to be *optimally* willing to give up 0.134 points of armor or magic resistance for each point of health he could gain.

Finally, we must construct a utility function $U(x, y) = U(\text{armor, health}) = U(\text{magic resistance, health})$ that describes the behavior of effective health given a bundle of armor or magic resistance and health. Since utility units are given in terms of effective health, $U(x, y)$ will calculate the amount of effective health given any bundle of armor or magic resistance and health. This utility function is relatively simple to understand since it is just the calculation of how much effective health armor and health will give:

$$U(x, y) = \frac{xy}{100} + x \text{ where } x \text{ is health and } y \text{ is armor}$$

Mathematically, this equation is equivalent to the above process of calculating effective health from armor/magic resistance and health.

Using this information, we will now solve the optimization problem. Now, we must calculate the Marginal Rate of Substitution for the utility function. Using calculus, the Marginal Rate of Substitution for our utility function is the negative ratio of the marginal utilities:

$$-\left[\frac{MU_x}{MU_y}\right] = -\left[\frac{\frac{\delta U(x,y)}{\delta x}}{\frac{\delta U(x,y)}{\delta y}}\right] = -\left[\frac{\frac{y}{100} + 1}{\frac{x}{100}}\right] = -\frac{y + 100}{x}$$

Now, we will perform the Lagrangean Multiplier Method to maximizing our utility function given our budget constraint. The budget constraint with prices included is:

$$B = 2.6x + 19.4y$$

For the Lagrangean Multiplier Method using a Lagrange multiplier λ, we write:

$$L = u(x, y) + \lambda \left[B - p_x x - p_y y\right]$$
$$L = \frac{xy}{100} + x + \lambda \left(B - 2.6x - 19.4y\right)$$

Differentiating L with respect to x, y, and λ gives us the first order conditions. The first order conditions are initially 0, but after some basic algebra, we get them in the following form (except the third which remains equal to 0):

$$\frac{\delta u}{\delta x} = 2.6\lambda \ , \ \frac{\delta u}{\delta y} = 19.4\lambda \ , \ \frac{\delta u}{\delta \lambda} = B - 2.6x - 19.4y = 0$$

Then, substituting in partial derivatives:

$$\frac{y + 100}{260} = \lambda \ , \ \frac{x}{1940} = \lambda$$
$$\frac{y + 100}{260} = \frac{x}{1940}$$
$$x = \frac{97(y + 100)}{13}$$

Now, we substitute this into the third first order condition to find an optimal amount of armor y at budget B:

$$B - 2.6 \left[\frac{97(y + 100)}{13}\right] - 19.4y = 0$$
$$B - 19.4y - 1940 - 19.4y = 0$$
$$B = 38.8y + 1940$$
$$\frac{B - 1940}{38.8} = y$$

Having solved for y, we can plug this in to the equation that gives x.

$$x = \frac{97(y + 100)}{13}$$
$$x = \frac{97((\frac{B-1940}{38.8}) + 100)}{13}$$

Now, we have shown that for some amount of gold B, the most optimal allocation of armor and health (or magic resistance and health) is:

$$\text{Health} = \frac{97((\frac{(B-1940)}{38.8}) + 100)}{13}$$

$$\text{Armor} = \frac{B - 1940}{38.8}$$

It is interesting to note that, until a player has more than 746 health (the solution of the Health equation for $B = 1,940$ gold), they should not spend gold on armor. After that point, though, the optimal purchase path is significantly more diverse between armor (magic resistance) and health. Understand that B is the *total* amount of gold a player spends on armor (magic resistance) and health in a game, *not* the amount of gold they have in certain situations.

3 A Modified Optimality Approach and the "Optimal Stats" Curve

As some of the closer readers may have noticed, though, optimal choices do not happen in a complete vacuum. Players do not start the game with B amount of gold with 0 health and 0 armor and magic resistance. Instead, they have base statistics that grant them initial armor, health, and magic resistance values. While gold can be used to purchase items which can supplement base values, the base values are essentially free.

How, then, can we modify our model to take this into account?

3.1 Adding Value to Base Statistics

Recall the Marginal Rate of Substitution MRS is the ratio of the per-unit prices of armor, health, and magic resistance.

$$\text{MRS} = \frac{p_x}{p_y} \approx 0.134$$

The interpretation of this value is "optimally, a player should be willing to give up 0.134 points of armor or magic resistance if he can gain 1 point of health." Taking the inverse of this value gives an approximate ratio of 7.46.

Shown this way, a player ought to trade 7.46 points of health per additional unit of armor or magic resistance.

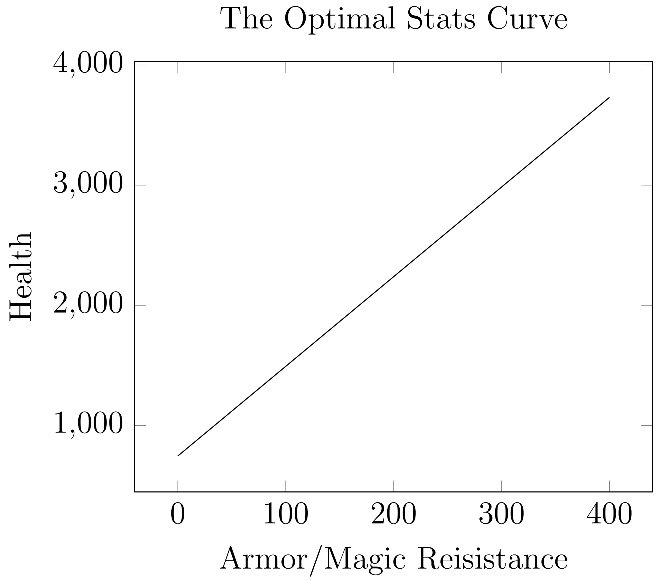
The optimality condition of economics states that the indifference curve (graph of the utility function) should be tangent to the budget constraint line with a slope of the MRS. Thus, from this, we can correct our model - now, given any level of health or armor (equivalently magic resistance), a player can find the best purchase given their levels of statistics. Given the above economic derivation, the following equation ought to perfectly maximize effective health:

$$y = 7.4615x + 746$$

for x = armor and y = health.

3.2 The Optimal Stats Curve

The Optimal Stats Curve is a graphical representation of optimal amounts of armor, health, and magic resistance. Its slope is the MRS of armor/magic resistance to health, and its intercept is at 746 health (remember, up until that point, it is actually *more* efficient to continue buying health).



From the Optimal Stats Curve, we see all of the amounts of armor and magic resistance a player should have at different health levels and vice versa to maximize the amount of effective health they have. Let us briefly examine the graph and draw some conclusions.

First, health seems to be a greatly overvalued resource. Buying 6 Warmog's Armors is highly inefficient. Even though the total health pool may seem larger, it is the armor and magic resistance that scale the health that provides value to the health resource. Builds that focus on health may be more effective against true or mixed damage. However, they are not optimal in terms of pure effective health maximization. Player's should very rarely, if ever, be significantly above $4,000$ health (an expensive proposition in terms of gold), especially if they do not have the armor to match it.

Second, from the author's personal experience, players never tend to have enough armor or undervalue armor at lower amounts of health. The Optimal Stats Curve clearly shows that even at relatively low health values, more armor generally means more effective health. If a player is rushing Randuin's Omen for their first purchase, it may actually be more useful to purchase the Warden's Mail first, especially if the player has Doran's items. As a quick example, at $2,000$ health, a player should already have 168 armor.

While there are undoubtedly more conclusions that can be drawn from the Optimal Stats Curve, the most key conclusion should be the relative levels of armor/magic resistance and health that will provide the most effective health. As recently as a few days ago, I witnessed a pro-level streamer state that he needed to buy as much armor as he could to mitigate the enemy team's preponderance of physical damage. The player was playing Hecarim and had $2,450$ health with 344 armor. They had one more item slot available. The player decided to purchase a Frozen Heart based on the misguided belief that it would provide him the most tanking power (the attack speed slow and mana were not considered particularly valuable). However, the Frozen Heart purchase was not in line with the Optimal Stats Curve. The optimal amount of health for 344 armor is $3,312$. Increasing the armor to 444 would require $4,058$ health to be optimal. Had the player bought the Warmog's Armor instead of the Frozen Heart, he would have had $14,430$ effective health instead of $13,328$ effective health, a difference of $1,102$. If the player had had that additional health, it may have allowed him to survive long enough to help his team win the game.

It is thus important for player's to understand the choices they make when buying in-game statistics. It can be very detrimental and a waste of gold to buy the wrong items instead of maximizing effective health. Proper gold allocation should be the first and foremost thing that players look to do in game - anything else is inefficient.

4 Notes and Limitations

This economic model should be taken with both a grain of salt and some knowledge of its limitations.

4.1 Item Costs

In League of Legends, it is not possible to buy individual statistics with gold. Instead, players are required to purchase items that come with many different statistics and effects. It should be noted, then, that while an optimal build path may be reached, it will never perfectly meet the budget constraint. Because of this, players should feel free to take the liberty of buying the item effects that they need. For example, if a tank feels like they need damage earlier in the game, the Sunfire Cape may be a good option compared to Randuin's Omen, especially on high base damage characters.

This, however, is not to say that player's should not try to be efficient in their buys. The Hecarim example above shows that little differences in itemization can have enormous effects in terms of efficiency of purchase. Players should look to buying the items that will keep them closest to the Optimal Stats Curve; the closer a player is to the curve, the more efficient their purchases are and the more effective health they have.

4.2 Slot Efficiency

Slot efficiency is defiend as the ability for higher tier items to offer many statistics in a single item slot than multiple lower tier items. If there were an infinite amount of item slots instead of 6, buying Doran's items and other tier 1 items would actually be more gold efficient than buying many of the higher tier items (disincluding things like item effects). This economic model could occasionally ask players to purchase odd combinations of lower tier items instead of higher tiered ones. Thus, a player may not have the room to accommodate many lower tier items and instead may be forced to upgrade their existing ones. A discussion of slot efficiency will follow in a later article.

4.3 A New Way of Analyzing Item Efficiency?

Currently, item efficiency is usually calculated using the amount of statistics an item gives compared to the average cost of buying the statistic. This "gold efficiency" approach assumes that items are already relatively optimal in their statistics distribution, and the only thing at issue is how much of a particular statistic an item has. However, the economic model established in this paper indicts the assumption that items are relatively efficient in their statistic distribution (amounts of armor, health, and magic resistance are optimal inside the item).

Instead, we will now briefly examine a new way of measuring item efficiency in terms of the efficiency of statistics distribution. If players ought to optimally give up 7.46 points of health for an additional point of armor/magic resistance, then an efficient item should have a ratio of statistics that relatively follow the Health and Armor/Magic Resistance equations for some price B. It is interesting to note that many "cost-efficient" items under the gold-efficiency model do not actually provide optimal levels of statistics for their price.

Items, by design, are not perfectly efficient. One reason may be that they are meant to supplement base statistics. For example, the Frozen Heart does not provide any health at all - it should be used as exclusively an armor supplement. Additionally, giving health to items like the Thornmail or Frozen Heart may make them "broken" so to speak - this speaks, once again, to the idea of slot efficiency.

Consider Randuin's Omen: at a price of $2,850$ gold, it should provide 921 health and 23 armor, excluding its passives in gold calculation. The item is 54% Health efficient and 298% armor efficient. Items that are closer to 100% efficient on both equations are theoretically better to spend gold on since they better maximize effective health.

The gold-efficiency model is useful for comparing one item directly to another. A player, following the Optimal Stats Curve, decides he needs an additional 100 armor. If his choices are Frozen Heart and Thornmail, he should go with the more gold-efficient of the two to maximize his slot efficiency. However, the gold-efficiency model is *not* a good indicator of overall efficiency of an item. The statistics distribution efficiency model allows a player to compare an item's efficiency overall.

4.4 The Binary Relationship Between Armor/Magic Resistance and Health

Earlier in the paper, we discussed how this model only treats armor, health, and magic resistance as binary relationships: in this model, the player only looks to purchase one or the other since the enemy team is exclusively one damage type. Obviously, this is not the case in most games players will experience. Additionally, our model previously assumed that players will neglect wards, damage items, and elixirs to focus exclusively on tanking statistics. This is not entirely true.

Players should look to relatively follow the Optimal Stats Curve regarding any amount of gold they spend on taking statistics. They can maximize their magic resistance and armor in relation to their health levels in two scenarios: 1) the enemy team is exclusively one damage type or 2) the enemy team's damage is perfectly mixed (in the latter scenario, players should buy exactly the same amount of armor and magic resistance and then buy the optimal amount of health). Further research will have to be performed to optimize ratios of armor, health, and magic resistance in cases where enemy teams have different damage types.

5 Conclusion

Throughout this article, we have attempted to show that optimal build paths in League of Legends exist, especially in regards to armor, health, and magic resistance. By utilizing the Lagrangean Multiplier Method to maximize the effective health utility function, we have sought to show that certain combinations of statistics are better than others when attempting to tank damage for a team. Every argument that applies to armor and health equivalently applies to magic resistance and health.

Of all the questions that League of Legends requires answers to, "how should I spend my gold?" has been a fundamental one. If limitations on gold did not exist, then players could select their late game inventory depending on their needs and the needs of their team. However, this is not the case. As such, it is necessary for players to maximize the value of the gold they spend in order to best facilitate winning.